Phase Change in a Radiating and Convecting Medium with Variable Thermal Properties

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One-dimensional heat transfer involving solidification of materials with temperature-dependent thermal properties subjected to radiation and convection is studied. In the present analysis, Biot's variational principle is employed. The complicated nonlinear heat transfer problem is reduced to a much simpler initial value problem which is then solved by standard techniques. The numerical results for the time variant solid liquid interface and the temperature history at the heat transfer surface are presented graphically.

Nomenclature

B $= L/c_{pr}T_i$ for solidifcation $= L/c_{pr}T_e$ for melting
B_0 $= h_0L_l/k_r =$ Biot number
c_p = specific heat
c_{pr} = a referenced specific heat obtained from a linear curve fitting or a pseudospecific heat at the absolute zero of temperature
D = dissipation function
F = thermal potential per unit volume
h = heat content per unit volume
h_0 = heat transfer coefficient
H = heat flow vector
$\dot{H}$ = local heat flux
k = thermal conductivity
k_r = a referenced conductivity obtained from a hyperbolic curve fitting or a pseudoconductivity at the absolute zero temperature
L = latent heat
L_l = thickness of slab
$\bar{M}$ = dimensionless heat capacity $\bar{M} = (\rho c_p)/(\rho c_p)_r$
M = coefficient for variable heat capacity defined by Eq. (7a)
n = unit vector outward normal to boundary surface
N = coefficient for variable conductivity defined by Eq. (7b)
$\bar{N}$ = dimensionless thermal conductivity, $\bar{N} = k/k_r$
N_r = radiation parameter
$= \sigma F T_i{}^3 L_l/k_r$ for solidification
$= \sigma F T_e{}^3 L_l/k_r$ for melting
p = phase change parameter, $p=1$ for solidification, $p=-1$ for melting
s = dimensionless solidification or melting distance, $s = \delta/L_l$
t = dimensional time
T = dimensional temperature
u = dimensionless temperature
$= T/T_i$ for solidification
$= T/T_e$ for melting
u_e = dimensionless environment temperature
$= T_e/T_i$ for solidification
$= 1$ for melting
u_p = dimensionless fusion or melting temperature
$= 1$ for solidification
$= T_p/T_e$ for melting
u_s = dimensionless surface temperature
$= T_s/T_i$ for solidification
$= T_s/T_e$ for melting
u_i = dimensionless initial temperature
$= 1$ for solidification
$= T_i/T_e$ for melting
x = dimensional distance
ρ = density
σ = Stefan-Boltzmann constant
α = thermal diffusivity, $k/\rho c_p$
δ = dimensional melting or solidification distance
τ = dimensionless time, $\alpha_r t/L_l^2$
θ = dimensionless temperature above the initial temperature $= u - u_i$
ξ = dimensionless distance, x/L_l

Subscripts

p = condition at melting or fusion temperature
i = initial condition ($i=p$ for the present case)
e = environment condition
s = condition at $x=0$

Introduction

PROBLEMS involving transient heat conduction with phase change are of considerable importance in many technical fields. Typical examples in the field of engineering are the aerodynamic heating of space vehicles, the making of ice, the freezing of foods, and the solidification in casting processes. The surface radiation becomes particularly important in the case of metal casting at high temperatures. Whatever the field of application, the problem of central interest is to determine how the melting or fusion propagates and how the temperature is distributed in the old and newly formed phases. Recently, the approximate analyses[1,2] were presented for phase change in a planar medium with the initial temperature either equal to or different from the fusion temperature. The analysis was based on the assumption that the thermal properties of the material are independent of temperature. This assumption is reasonable for modest temperature ranges. However, when the temperature difference is large, the effect of variable properties becomes significant. The purpose of this work is to extend the previous study[1] by including the effect of variable properties. As a result of the strong nonlinearity of the problem, an exact solution is not feasible. In the present analysis, Biot's variational method[3-6] is employed with an emphasis on the solidification of saturated liquids. With this method the complicated nonlinear boundary value problem is reduced to a much simpler initial value problem which can be easily solved using standard numerical techniques.

Mathematical Analysis

Consideration is given to a finite region of liquid (or solid) which is initially at its fusion or melting temperature, T_p. Due to radiative and aerodynamic cooling (or heating) at one surface, phase change takes place, while the other side is ther-

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mally insulated. It is assumed that during the period of phase change, there exists, at all times, a sharply-defined line of division between the solid and liquid phases. The thermal properties except density are temperature dependent. A constant density is assumed for both liquid and solid phases. The justification of this assumption may be found in the literature[7,8] which have demonstrated that the effect of unequal phase densities is negligibly small. Furthermore, heat transfer due to free convection in the liquid region is assumed negligibly small and the newly formed phase is assumed to be opaque.

Since the material under consideration is initially at fusion (or melting) temperature, the old phase always maintains a constant temperature at u_p. The governing equations for the newly formed phase in terms of dimensionless variables and parameters are as follows

$$\bar{M}\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial \xi}\left(\bar{N}\frac{\partial u}{\partial \xi}\right) \qquad 0 \le \xi \le s \tag{1}$$

$$u = u_i = u_p \qquad \text{at } \tau = 0,\ \xi > 0 \tag{2a}$$

$$\bar{N}\frac{\partial u}{\partial \xi} = N_r(u^4 - u_e^4) + B_0(u - u_e) \qquad \text{at } \xi = 0,\ \tau > 0 \tag{2b}$$

$$\bar{N}\frac{\partial u}{\partial \xi} = pB\frac{\mathrm{d}s}{\mathrm{d}t} \qquad \text{at } \xi = s,\ \tau > 0 \tag{2c}$$

Where s is dimensionless solidification or melting distance; $\bar{M}$ and $\bar{N}$ are dimensionless heat capacity and conductivity respectively. $p=1$ represents the case of solidification, $p=-1$ for melting.

To apply Biot's variational principle, we first define the heat content per unit volume, h, and the thermal potential per unit volume, F, as[4]

$$h = \int_0^\theta \bar{M}(u)\,\mathrm{d}u \tag{3}$$

$$F = \int_0^\theta \bar{M}(u)u\,\mathrm{d}u \tag{4}$$

Where θ is the temperature field above the equilibrium temperature. We then define a heat flow vector, $\boldsymbol{H}$, whose time rate of change, $\dot{\boldsymbol{H}}$, represents the heat flux across the area normal to $\boldsymbol{H}$. From energy balance and divergence theorem, we obtain

$$\text{div}\ \boldsymbol{H} = -h \tag{5}$$

Based on the variational principle, Biot has derived the following useful equation of Lagrangian type[3]

$$\frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = -\int_A \theta\frac{\partial \boldsymbol{H}}{\partial q_i}\cdot \boldsymbol{n}\mathrm{d}A \tag{6}$$

Where q_i is the generalized coordinates, $\boldsymbol{n}$ is the unit vector outward normal to the boundary A of the body. V is the thermal potential defined by

$$V = \int_v F\,\mathrm{d}V$$

D is the dissipation function defined by

$$D = \frac{1}{2}\int_v \frac{1}{\bar{N}(u)}\dot{H}^2\,\mathrm{d}v$$

and v is the volume of the body.

For convenience, let the heat capacity and thermal conductivity be the forms

$$\bar{M} = 1 + Mu \tag{7a}$$

$$\bar{N} = \frac{1}{1+Nu} \tag{7b}$$

We assume a linear temperature profile in the newly formed phase

$$\theta = (u_s - u_p)\,[1 - (\xi/s)] \tag{8}$$

where θ is the temperature above the initial temperature, ($u_i = u_p$ for the present case). Note that the choice of linear temperature profile in the newly-formed phase has been shown to yield quite satisfactory results in the earlier studies[9-11] The heat flow vector is obtained from Eqs. (5, 7, and 8)

$$H = \frac{1}{2}(1 + Mu_p)(u_s - u_p)s\left(1 - \frac{\xi}{s}\right)^2 + \frac{1}{6}M(u_s - u_p)^2 s\left(1 - \frac{\xi}{s}\right)^3 - PBs \tag{9}$$

Substituting Eqs. (7-9) into Eq. (6) leads to a first-order ordinary differential equation in terms of u_s and s of the form[12]

$$D_1 + D_2 + D_3 + D_4 + D_5 = \frac{1}{3}(1 + Mu_p)(u_s - u_p)^2 + \frac{1}{12}M(u_s - u_p)^3 - PB(u_s - u_p) \tag{10}$$

where

$$D_1 = \frac{3}{40}a_1^2 b_1 c_1 s(\dot{s}c + s\dot{u}_s) + \frac{1}{45}Ma_1 b_1 c_1^2 s(\dot{s}c_1 + s\dot{u}_s) + \frac{7}{120}$$
$$Na_1^2 c_1^2 s(\dot{s}c_1 + s\dot{u}_s) + \frac{1}{56}MNa_1 c_1^3 s(\dot{s}c_1 + s\dot{u}_s) - \frac{1}{6}PBs$$
$$a_1 b_1(\dot{s}c_1 + s\dot{u}_s) - \frac{1}{8}NPBs\ a_1 c_1(\dot{s}c_1 + s\dot{u}_s)$$

$$D_2 = \frac{7}{120}a_1^2 b_1 c_1^2 s\dot{s} + \frac{1}{72}a_1 b_1 c_1^3 Ms\dot{s} + \frac{1}{30}Na_1^2 c_1^3 s\dot{s}$$
$$+ \frac{11}{1260}MNa_1 c_1^4 \dot{s}s - \frac{1}{6}a_1 b_1 c_1 PBs\dot{s} - \frac{1}{12}Na_1 c_1^2 PBs\dot{s}$$

$$D_3 = \frac{7}{360}Ma_1 b_1 c_1^2(\dot{s}c_1 + 2s\dot{u}_s)s + \frac{1}{168}M^2 b_1 c_1^3(\dot{s}c_1 + 2s\dot{u}_s)s$$
$$+ \frac{1}{63}MNa_1 c_1^3(\dot{s}c_1 + 2s\dot{u}_s)s + \frac{5}{1008}M^2N\ c_1^4(\dot{s}c_1 + 2s\dot{u}_s)s$$
$$- \frac{1}{24}Mb_1 c_1(\dot{s}c_1 + 2s\dot{u}_s)PBs - \frac{1}{30}MNc_1^2 PBs(\dot{s}c_1 + 2s\dot{u}_s)$$

$$D_4 = \frac{1}{60}Ma_1 b_1 c_1^3 \dot{s}s + \frac{11}{2520}M^2 b_1 c_1^4 \dot{s}s + \frac{3}{280}MNa_1 c_1^4 \dot{s}s$$
$$+ \frac{79}{24864}M^2 Nc_1^5 s\dot{s} - \frac{1}{24}Mb_1 c_1^2 PBs\dot{s} - \frac{1}{40}MNc_1^3 PBs\dot{s}$$

$$D_5 = -\frac{1}{3}a_1b_1c_1PBs\dot{s} - \frac{1}{12}Mb_1c_1^2PBs\dot{s} - \frac{5}{24}Na_1c_1^2PBs\dot{s}$$

$$-\frac{7}{120}MNc_1^3PBs\dot{s} + B^2b_1s\dot{s} + \frac{1}{2}B^2Nc_1s\dot{s}$$

$$a_1 = 1 + Mu_p$$

$$b_1 = 1 + Nu_p$$

$$c_1 = u_s - u_p$$

$$\dot{s} = \frac{\mathrm{d}s}{\mathrm{d}\tau}$$

$$\dot{u}_s = \frac{\mathrm{d}u_s}{\mathrm{d}\tau}$$

and

$$\tau = \alpha_r t / L_1^2$$

The variable s in Eq. (10) can be eliminated by introducing the following relationship

$$s = -\frac{u_s - u_p}{(1 + Nu_s)\,[N_r(u^4 - u_e^4) + B_0(u_s - u_e)]} \tag{11}$$

which is obtained from the boundary condition at $\xi = 0$. Therefore, we obtain an equation for the surface temperature of the form $\dot{u}_s = f(u_s, \tau)$ with initial condition, $u_s(0) = 1$. This first-order differential equation can be easily solved numerically using the available subroutine in an IBM scientific package.

Results and Discussion

Although the present analysis is applicable to both melting and freezing, the numerical computations are presented for the case of solidification only, i.e., $p = 1$. The same analysis can be equally applied to solidification of a semi-infinite region of liquid. Furthermore, similar techniques can be employed in the spherical and cylindrical systems involving a phase change. Figures 1 and 2 show the effects of variation of thermal conductivity upon the rate of solidification and the radiative surface temperature under various values of B. The effects due to the variation of heat capacity are presented in Figs. 3-4. The parameters involved in these figures are $B_0 = 1$, $N_r = 1$ and $u_e = 0.25$. It is found that the constant properties solution fall between the curves with $N = \pm 0.5$, $M = 0$, or $M = \pm 0.5$, $N = 0$. All curves end at the instant when s equals unity. As expected, the total solidification time increases as B increases. The radiative surface temperature increases when the heat capacity increases or when the thermal conductivity increases. Figures 2 and 3 also show that the surface temperature increases as B increases at a given time, because it takes more time to cool the surface is the latent heat is higher or heat capacitance is lower.

It is seen from Figs. 1 and 4 that the solidification front moves faster when either the thermal conductivity is larger or the heat capacity is smaller, since the high conductivity or low heat capacity of the material tends to accelerate the heat transfer process. Consequently, the total solidification time is reduced. The time required to reach a specific value of s increases as the value of B increases. It is of interest to note from Figs. 3 and 4 that the effect of the variation of the heat capacity on u_s and s becomes less and less important when B becomes larger. The reason is that the rate of solidification, $\mathrm{d}s/\mathrm{d}\tau$ heavily depends on B when B is very large. However, the surface temperature and the solidification distance are affected greatly by the variation of the thermal conductivity with respect to temperature.

Figures 5 and 6 illustrate the surface temperature history and the time variant fusion line for variable heat capacity and thermal conductivity of the material. It is seen that the surface temperature of the material with $M = -0.5$ and $N = 0.5$ cools

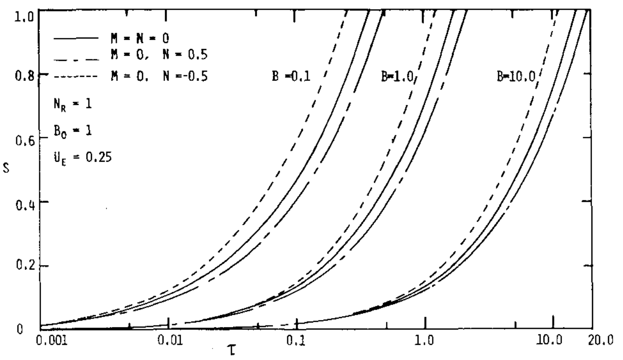

Fig. 1 Solidification front movement with time for variable thermal conductivity.

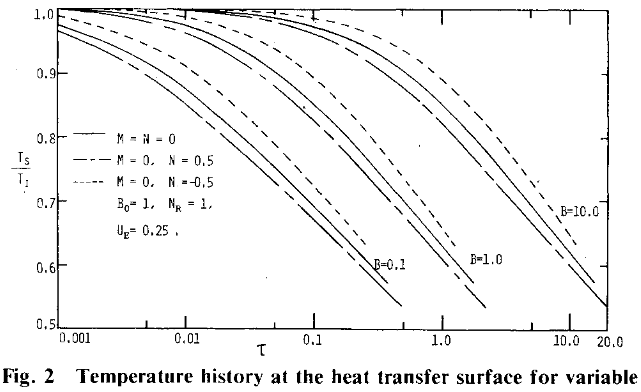

Fig. 2 Temperature history at the heat transfer surface for variable thermal conductivity.

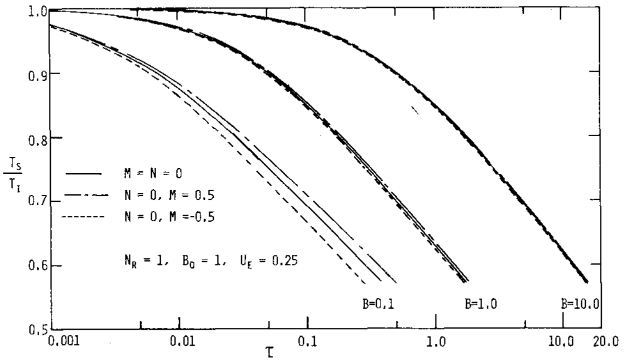

Fig. 3 Temperature history at the heat transfer surface for variable heat capacity.

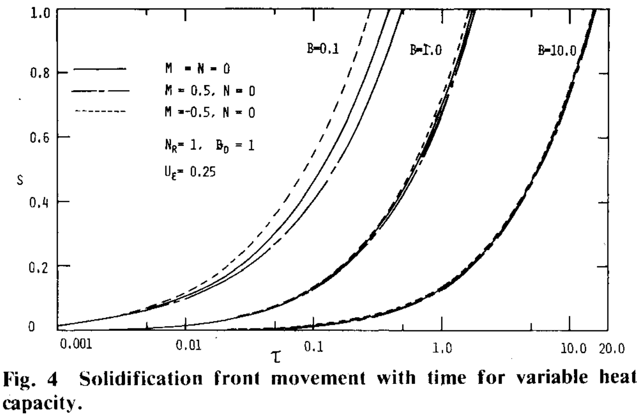

Fig. 4 Solidification front movement with time for variable heat capacity.

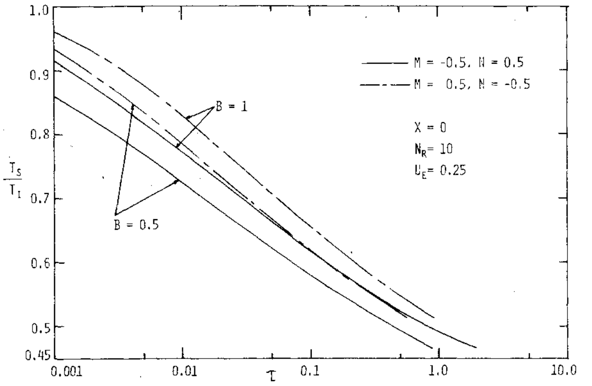

Fig. 5 Temperature history of the heat transfer surface with variable heat capacity and thermal conductivity.

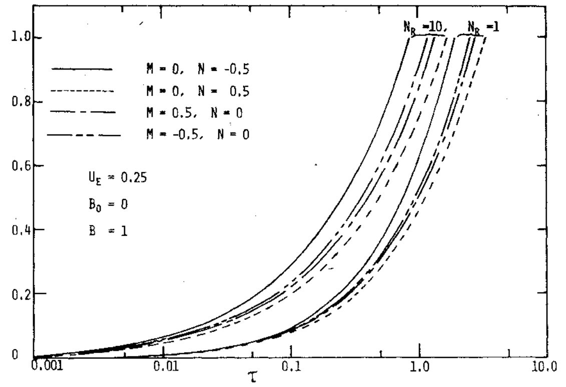

Fig. 7 Solidification front movement with time for either variable heat capacity or thermal conductivity.

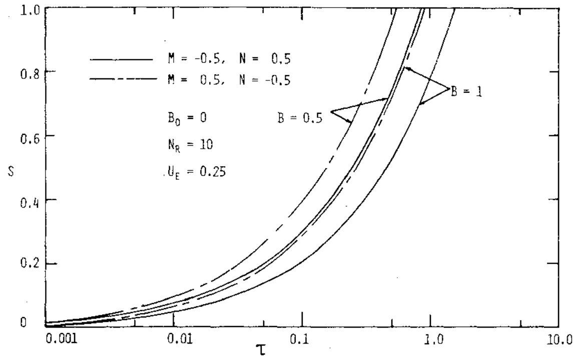

Fig. 6 Solidification front movement with time for variable thermal conductivity and heat capacity.

faster than that of the case with $M=0.5$ and $N=-0.5$ during solidification. Figure 7 plots the interfacial position against the dimensionless time with radiation numbers as a parameter at different values of M and N. It is seen for these particular parameters that the curves with $N=0$ fall between those with $M=0$ at the same radiation number. It appears that a variable conductivity has more effect than a correspondingly specific heat. At a given time, the difference of s between two curves with $M=0$, $N=0.5$, and $M=0$, $N=-0.5$ increases as N_r increases. A similar trend is found for the case with $N=0$, $M=\pm 0.5$.

It should be pointed out that representations of $\bar{M}$ and $\bar{N}$ given by Eq. (7) are reasonable approximations for a large number of metals and alloys. Although Eqs. (7a) and (7b) are not applicable to very low temperature region of metals, they cover a wide temperature range up to fusion temperature. Note that the values of ρc_{pr} and k_r do not represent the true values of heat capacitance and conductivity at the temperature of absolute zero. (The theoretical value should be zero). They only represent the extrapolation values at the absolute zero based on the least squares curve fitting to the experimental data within a given temperature range. For illustrative purposes Table 1 lists recommended values of M, c_{pr}, N, k_r and B for some materials encountered in metal casting. The first four items are computed based on the least squares curve fitting from the experimental data compiled by Goldsmith et al.[13]

Due to the general nature of the present problem, no previous solution is available for comparison. However, as indicated before, a special case with constant properties has been studied by these authors.[1] We now consider the case that material properties are temperature independent. The solution for solidification depth and surface temperature can be directly obtained by letting $M=N=0$, Eqs. (10) and (11) yield

$$B^2 s\frac{\mathrm{d}s}{\mathrm{d}\tau} - \frac{2}{3}PB(u_s-u_i)s\frac{\mathrm{d}s}{\mathrm{d}\tau} + \frac{2}{15}(u_s-u_i)^2 s\frac{\mathrm{d}s}{\mathrm{d}\tau}$$

$$+\frac{3}{40}s^2(u_s-u_i) - \frac{1}{6}PBs^2\frac{\mathrm{d}u_s}{\mathrm{d}\tau} = \frac{1}{3}(u_s-u_i)^2 - PB(u_s-u_i)$$

which is identical to Eq. (12) of Ref. 1.

Conclusions

Analysis of one-dimensional heat transfer involving solidification of a planar medium with variable thermal properties subjected to radiation and convection are presented. Biot's variational method is employed to the solution of this complicated nonlinear problem. It is found that the depth of solidification or melting increases as the radiation number increases. The depth of solidification and the radiative surface temperature are found to be not quite sensitive to the variation of heat capacity when B is very large. However, the radiative surface temperature and solidification distance are greatly affected by the variation of thermal conductivity. It is also found that total solidification time increases as B increases. A limiting solution that the material has constant properties is directly obtained from the present solution by letting $M=N=0$.

References

[1] Chung, B. T. F. and Yeh, L. T., "Solidiation and Melting of Materials Subject to Convection and Radiation," *Journal of Spacecraft and Rockets*, Vol. 12, June 1975, pp. 329-333.

Table 1 Recommended values of c_{pr}, k_r, M, N and B for some metals

Element	Temperature range, °R	c_{pr}, BTU/lb$_m$-°R	k_r, BTU/hr-ft-°R	M	N	B[a] $L/c_{pr}T_i$
Aluminum	350-1678	0.180	150.9	0.667	0.486	0.563
Copper	350-2440	0.850	256.6	0.432	0.424	0.289
Magnesium	350-1661	0.201	83.6	0.603	0.122	0.473
Silver	500-1740	0.0523	295.8	0.286	0.735	0.344

[a] In Ref. 1, B is defined by $L/c_p T_i$ where c_p is based on fusion temperature.

[2] Yeh, L.T. and Chung, B.T.F., "Transient Heat Conduction in a Finite Medium with Phase Change," ASME Paper 76 WA/HT-3, 1976.

[3] Biot, M. A., "New Methods in Heat Flow Analysis with Application to Flight Structures," *Journal of the Aeronautical Sciences*, Vol. 24, 1957, p. 857.

[4] Biot, M. A., "Further Developments of New Methods in Heat Flow Analysis," *Journal of the Aeronautical Sciences*, Vol. 26, June 1959, p. 367.

[5] Lardner, T. J., "Biot's Variational Principle in Heat Conduction," *AIAA Journal*, Vol. 1, Jan. 1963, pp. 196-206.

[6] Chung, B. T. F, and Yeh, L. T., "Analysis of Heat Transfer in Slabs with Variable Properties Subjected to Radiation and Convection," *ASME Paper 75-WA/HT-67*, 1975.

[7] Heitz, W. L. and Westwater, J. W., "Extension of the Numerical Method for Melting and Freezing Problems," *International Journal of Heat and Mass Transfer*, Vol. 13, 1970, p. 1371.

[8] Bonacina, C., Comini, G., Fasano, A. and Primicerio, M., "Numerical Solution of Phase Change Problems," *International Journal of Heat and Mass Transfer*, Vol. 16, 1973, p. 1825.

[9] Prasad, A. and Agrawal, H. C., "Biot's Variational Principle for a Stefan Problem," *AIAA Journal*, Vol. 10, March 1972, pp. 325-327.

[10] Prasad, A. and Agrawal, H. C., "Biot's Variational Principle for Aerodynamic Ablation of Melting Solids," *AIAA Journal*, Vol. 12, Feb. 1974, pp. 250-252.

[11] Hrycak, P., "Problem of Solidification with Newton's Cooling at the Surface," *AIChE Journal*, Vol. 8, 1963, p. 585.

[12] Yeh, L. T., "On Biot's Variational Principle for Some Transient Heat Transfer Processes Involving Thermal Radiation," Ph.D. Disertation, Department of Mechanical Engineering, The University of Akron, Akron, Ohio, 1976.

[13] Goldsmith, A., Waterman, T. E. and Hirschhorn, H., *Handbook of Thermophysical Properties of Solid Materials*, Macmillan, New York, 1961.